

# MINIMAL GRADED FREE RESOLUTIONS FOR MONOMIAL CURVES DEFINED BY ARITHMETIC SEQUENCES

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**ABSTRACT.** Let  $\mathbf{m} = (m_0, \dots, m_n)$  be an arithmetic sequence, i.e., a sequence of integers  $m_0 < \dots < m_n$  with no common factor that minimally generate the numerical semigroup  $\sum_{i=0}^n m_i \mathbb{N}$  and such that  $m_i - m_{i-1} = m_{i+1} - m_i$  for all  $i \in \{1, \dots, n-1\}$ . The homogeneous coordinate ring  $\Gamma_{\mathbf{m}}$  of the affine monomial curve parametrically defined by  $X_0 = t^{m_0}, \dots, X_n = t^{m_n}$  is a graded  $R$ -module where  $R$  is the polynomial ring  $k[X_0, \dots, X_n]$  with the grading obtained by setting  $\deg X_i := m_i$ . In this paper, we construct an explicit minimal graded free resolution for  $\Gamma_{\mathbf{m}}$  and show that its Betti numbers depend only on the value of  $m_0$  modulo  $n$ . As a consequence, we prove a conjecture of Herzog and Srinivasan on the eventual periodicity of the Betti numbers of semigroup rings under translation for the monomial curves defined by an arithmetic sequence.

## INTRODUCTION

The study of affine and projective monomial curves has a long history beginning with the classification of space monomial curves in [4]. One of the most dramatic results in the subject is the fact that the number of generators for the defining ideal of these curves in the affine space  $\mathbb{A}^{n+1}$  is unbounded, [1]. Much of the study to date has focussed on determining the generators and the first Betti number of the defining ideal for many different classes of monomial curves. In this paper, we study the later Betti numbers as well as the structure of the resolution. Exact generators in the case of curves defined by an arithmetic sequence or an almost arithmetic sequences are known; see [8], [7], [6]. In the case of arithmetic sequences, these ideals have another interesting structure as sum of two determinantal ideals; see [3]. This provides the main impetus for understanding the resolution of these ideals. In this article, we construct the minimal resolution explicitly for these ideals and compute all the Betti numbers.

The main goal of this article is to prove the following conjecture that states that in codimension  $n$ , there are exactly  $n$  distinct patterns for the minimal graded free resolution of a monomial curve defined by an arithmetic sequence:

*Curves in affine  $(n+1)$ -space defined by a monomial parametrization  $X_0 = t^{m_0}, \dots, X_n = t^{m_n}$  where  $m_0 < \dots < m_n$  are positive integers in arithmetic progression have the property that their Betti numbers are determined solely by  $m_0$  modulo  $n$ .*

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Basis for this conjecture came from the observations in [11], where an explicit minimal free resolution was constructed for  $n = 3$ , using Gröbner basis techniques. Subsequently, this property was verified to hold for several examples in [12], and it was proved for certain special cases in [3], leading us to frame it as a conjecture. We give a complete proof of the conjecture in this article.

Let  $k$  denote an arbitrary field and  $R$  be the polynomial ring  $k[X_0, \dots, X_n]$ . Associated to a sequence of positive integers  $\mathbf{m} = (m_0, \dots, m_n)$ , we have the  $k$ -algebra homomorphism  $\varphi : R \rightarrow k[t]$  given by  $\varphi(X_i) = t^{m_i}$  for all  $i = 0, \dots, n$ . The ideal  $\mathcal{P} := \ker \varphi \subset R$  is the defining ideal of the monomial curve in  $\mathbb{A}_k^{n+1}$  given by the parametrization  $X_0 = t^{m_0}, \dots, X_n = t^{m_n}$  which we denote by  $C_{\mathbf{m}}$ . The  $k$ -algebra of the numerical semigroup  $\Gamma(\mathbf{m})$  generated by  $\mathbf{m} = (m_0, \dots, m_n)$  is the semigroup ring  $k[\Gamma(\mathbf{m})] := k[t^{m_0}, \dots, t^{m_n}] \simeq R/\mathcal{P}$  which is one-dimensional. Moreover,  $\mathcal{P}$  is a perfect ideal of codimension  $n$  and it is well known that it is minimally generated by binomials.

Since for any positive integer  $t$  the curve  $C_{t\mathbf{m}}$  is isomorphic to the curve  $C_{\mathbf{m}}$  for they have the same defining ideal, we may as well assume without loss of generality that the integers  $m_0, m_1, \dots, m_n$  have *no common factor*. Further, if the semigroup generated by a proper subset of  $\mathbf{m}$  equals the semigroup  $\Gamma(\mathbf{m})$ , then the curve  $C_{\mathbf{m}}$  is degenerate and the resolution of its coordinate ring can be studied in a polynomial ring with less variables. Hence we can reduce to the case where the integers in  $\mathbf{m}$  *minimally generate* the numerical semigroup  $\Gamma(\mathbf{m})$ . If, in addition, the integers  $m_i$  are in *arithmetic progression*, i.e.,  $m_i - m_{i-1} = m_{i+1} - m_i$  for all  $i \in \{1, \dots, n-1\}$ ,  $\mathbf{m}$  is said to be an **arithmetic sequence**.

An interesting feature that was revealed in [3] is that when  $\mathbf{m} = (m_0, \dots, m_n)$  is an arithmetic sequence, the ideal  $\mathcal{P}$  can be written as a sum of two determinantal ideals,  $\mathcal{P} = I_2(A) + I_2(B)$ , as we shall recall in Section 1.1. Here,  $I_2(A)$  is in fact the defining ideal of the rational normal curve in  $\mathbb{P}^n$ . Let us write  $m_0$  as  $m_0 = an + b$  where  $a, b$  are positive integers and  $b \in [1, n]$ . When  $b = 1$ , the sum  $I_2(A) + I_2(B)$  is again a determinantal ideal and its resolution is described in [3, Theorem 2.1 & Corollary 2.3]. When  $b = n$ , it is just one generator away from a determinantal ideal which is again simple; see [3, Theorem 2.4 & Corollary 2.5]. The case  $b = 2$  corresponds to the case where  $k[\Gamma(\mathbf{m})]$  is Gorenstein. The resolution was described in this case when  $n = 4$  in [3, Theorem 2.6]. This result will be completed in Section 2 where the resolution will be given for an arbitrary  $n$ .

The observation that the ideals  $I_2(A)$  and  $I_2(B)$  are related by linear quotients (Lemma 3.1) holds the key for the construction of the resolution of  $\mathcal{P}$  in general. We construct a tower of mapping cones each of which is a cone over an inclusion of a shifted graded Koszul complex into a graded Eagon-Northcott complex. Unfortunately, the above construction of iterated mapping cone does not yield a minimal free resolution for  $\mathcal{P}$  and therefore we will have to get rid of the redundancy and make the resolution minimal. The complete graded description of the resolution is given in the main Theorem 3.10. As a consequence, we compute the total Betti numbers  $\beta_j$  in Theorem 4.1 as follows: if  $m_0 \equiv b \pmod n$  with  $b \in [1, n]$ , then

$$\beta_j = j \binom{n}{j+1} + \begin{cases} (n-b+2-j) \binom{n}{j-1} & \text{if } 1 \leq j \leq n-b+1, \\ (j-n+b-1) \binom{n}{j} & \text{if } n-b+1 < j \leq n. \end{cases}$$

These numbers clearly depend only on the remainder of  $m_0$  modulo  $n$ , as conjectured in [3, Conjecture 1.2].

As another application of Theorem 3.10, we prove the following conjecture of Herzog and Srinivasan for monomial curves defined by an arithmetic sequence. The strong form of the conjecture says that if  $\mathbf{m}$  is any increasing sequence of non negative integers and  $\mathbf{m} + (j)$  denotes the sequence translated by  $j$ , then the Betti numbers of the semigroup ring  $k[\Gamma(\mathbf{m} + (j))]$  are eventually periodic in  $j$ . We prove in Theorem 4.11 that if  $\mathbf{m}$  is an arithmetic sequence, the strong form of the conjecture holds by showing that the betti numbers are periodic with period  $m_n - m_0 = nd$  where  $d$  is the common difference.

We will begin with some preliminaries on the defining ideal of monomial curves associated to arithmetic sequences, followed by some facts from mapping cones that we need in the paper. Section 2 is entirely on Gorenstein monomial curves defined by an arithmetic sequence where we construct the minimal graded free resolution of its coordinate ring as a direct sum of a resolution and its dual. Section 3 contains the construction of the minimal graded free resolution of the monomial curves defined by an arithmetic sequence in general. The last section has consequences of the main theorem (Theorem 3.10) for some relation between the regularity of the semigroup ring and the Frobenius number of the semigroup, an independent proof of the characterization of Gorenstein curves defined by an arithmetic sequence as well as the proof of the periodicity conjecture of Herzog and Srinivasan for the semigroup rings defined by an arithmetic sequence.

## 1. PRELIMINARIES

Let  $(\mathbf{m}) = (m_0, \dots, m_n)$  be an *arithmetic sequence*, i.e., a sequence of nonnegative integers such that  $m_i = m_0 + id$  for some  $d \geq 1$  and all  $i \in [0, n]$ , and such that  $m_0, \dots, m_n$  are relatively prime and minimally generate the numerical semigroup  $\Gamma(\mathbf{m}) := \sum_{0 \leq i \leq n} m_i \mathbb{N}$ . Note that one can always write  $m_0$  uniquely as

$$m_0 = an + b$$

with  $a, b$  positive integers and  $b \in [1, n]$ . The integer  $a$  is non-zero because the sequence  $(\mathbf{m}) = (m_0, \dots, m_n)$  is minimal.

### 1.1. Defining ideal of monomial curves associated to arithmetic sequences.

One knows by [3, Theorem 2.1] that the defining ideal  $\mathcal{P}$  of the affine monomial curve  $C_{\mathbf{m}} \subset \mathbb{A}_k^{n+1}$  is  $I_2(A) + I_2(B)$ , the sum of two determinantal ideals of maximal minors with

$$A = \begin{pmatrix} X_0 & \cdots & X_{n-1} \\ X_1 & \cdots & X_n \end{pmatrix}, \quad B = \begin{pmatrix} X_n^a & X_0 & \cdots & X_{n-b} \\ X_0^{a+d} & X_b & \cdots & X_n \end{pmatrix}.$$

It is well-known that the ideal  $I_2(A)$  is the defining ideal of the rational normal curve in  $\mathbb{P}_k^n$  of degree  $n$ ; see, e.g., [2, Proposition 6.1]. The fact that  $I_2(A)$  is contained in  $\mathcal{P}$  says that the affine monomial curve  $C_{\mathbf{m}}$  is lying on the affine cone over the rational normal curve. Indeed, the following easy lemma states that arithmetic sequences are precisely the ones whose associated monomial curve lies on this cone.

They are also the only sequences that make the ideal  $I_2(A)$  homogeneous with respect to the gradation obtained by setting  $\deg X_i = m_i$  for all  $i \in [0, n]$ .

**Lemma 1.1.** *Let  $\mathcal{P} \subset k[X_0, \dots, X_n]$  be the defining ideal of the non-degenerate monomial curve  $C_{\mathbf{m}} \subset \mathbb{A}_k^{n+1}$  associated to a strictly increasing sequence of integers  $\mathbf{m} = (m_0, \dots, m_n)$ . The following are equivalent:*

- (1)  $\exists d \in \mathbb{Z}$ , such that  $m_i = m_0 + id$ ,  $\forall i \in [0, n]$ ;
- (2)  $I_2(A) \subset \mathcal{P}$ ;
- (3)  $I_2(A)$  is homogeneous w.r.t. the weighted gradation on  $R$  given by  $\deg X_i = m_i$  for all  $i \in [0, n]$ .

*Proof.* As we already recalled, if  $m_i = m_0 + id$  for some  $d \geq 1$  then  $I_2(A) \subset \mathcal{P}$  and  $I_2(A)$  is homogeneous w.r.t. the weighted gradation, so (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3). Conversely, assuming that either (2) or (3) holds, one has that the integers in the sequence  $(\mathbf{m})$  satisfy that  $m_i + m_{j+1} = m_{i+1} + m_j$  for all  $i, j$  such that  $0 \leq i < j \leq n-1$ . In particular, for all  $j \in [1, n-1]$ , one has that  $m_0 + m_{j+1} = m_1 + m_j$ , i.e.,  $m_{j+1} = m_j + d$  if one sets  $d := m_1 - m_0 \geq 1$ .  $\square$

In this paper, homogeneous and graded will mean homogeneous and graded with respect to the weighted gradation on  $R$  given by  $\deg x_i = m_i$  for all  $i \in [0, n]$ . By Lemma 1.1,  $I_2(A)$  is homogeneous and  $\mathcal{P}$  is also homogeneous since  $\mathcal{P} := \ker \varphi$  and the map  $\varphi : R \rightarrow k[t]$  given by  $\varphi(X_i) = t^{m_i}$  is graded of degree 0.

**1.2. The weighted graded version of the Eagon-Northcott complex.** The minimal resolution of  $R/I_2(A)$  is given by the Eagon-Northcott complex of the matrix  $A$  because the height of  $I_2(A)$  is  $n$ .

Let  $F = \bigoplus_{i=1}^n R e_i$  be a free  $R$ -module of rank  $n$  with basis  $e_1, \dots, e_n$  and  $G = R g_1 \oplus R g_2$  be a free  $R$ -module of rank 2. The  $2 \times n$  matrix  $A$  represents a map  $\varphi : F \rightarrow G^*$ . The Eagon-Northcott complex  $\mathbf{E}$  of the matrix  $A$  is

$$\mathbf{E} : 0 \rightarrow E_{n-1} \xrightarrow{d_{n-1}} E_{n-2} \xrightarrow{d_{n-2}} \dots \rightarrow E_1 \xrightarrow{d_1} E_0,$$

with  $E_0 = R$  and, for all  $i \in [1, n-1]$ ,  $E_i := \wedge^{i+1} F \otimes D_{i-1} G$  and  $d_i$  is given by the diagonalization of  $\wedge F$  and multiplication of  $DG$ .

*Remark 1.2.* Note that  $R/I_2(A)$  is Cohen-Macaulay and the complex  $\mathbf{E}^*$  is exact. Thus, we get

$$\mathbf{E}^* : 0 \rightarrow R \xrightarrow{d_1^*} E_1^* \rightarrow \dots \xrightarrow{d_{n-2}^*} \wedge^{n-1} F^* \otimes D_{n-3} G^* \xrightarrow{d_{n-1}^*} \wedge^n F^* \otimes D_{n-2} G^*.$$

One has that  $d_1^* : R \rightarrow \wedge^2 F^*$  is the map  $\wedge^2(\varphi^*)$ , and for  $i > 1$ , the map  $d_i^* : \wedge^i F^* \otimes D_{i-2} G^* \rightarrow \wedge^{i+1} F^* \otimes D_{i-1} G^*$  is given by

$$d_i^*(x \otimes y) = \sum_{t=1}^n x \wedge e_t^* \otimes \varphi(e_t) y.$$

After identifying  $\wedge^n F^*$  with  $R$  and  $\wedge^{n-1} F^*$  with  $F$ , we see that

$$d_{n-1}^*(e_i \otimes g_1^{*(r)} g_2^{*(s)}) = (-1)^{n-i} [X_{i-1} g_1^{*(r+1)} g_2^{*(s)} + X_i g_1^{*(r)} g_2^{*(s+1)}].$$

This will be useful in Section 2.

The ideal  $I_2(A)$  is homogeneous with respect to the usual grading on  $R$  and the Eagon-Northcott complex is indeed a minimal graded free resolution of  $R/I_2(A)$ . This minimal graded resolution is 2-linear and it is as follows:

$$\begin{aligned} \mathbf{E} : 0 \rightarrow R^{n-1}(-n) \xrightarrow{d_{n-1}} R^{(n-2)\binom{n}{n-1}}(-n+1) \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_s} R^{(s-1)\binom{n}{s}}(-s) \xrightarrow{d_{s-1}} \cdots \\ \cdots \xrightarrow{d_2} R^{(n)}(-2) \xrightarrow{d_1} R \rightarrow R/I_2(A) \rightarrow 0. \end{aligned}$$

But  $I_2(A)$  is also homogeneous with respect to our weighted gradation on  $R$  as observed in Lemma 1.1 and the Eagon-Northcott complex is also a minimal graded free resolution of  $R/I_2(A)$  with respect to this weighted gradation. Of course, syzygies are no longer concentrated in one single degree at each step of the resolution as before. As observed in [3], the successive graded free modules in this resolution are  $E_0 = R$  and, for all  $s \in [2, n]$ ,

$$\begin{aligned} E_{s-1} &= \bigoplus_{1 \leq r_1 < \cdots < r_s \leq n} \left( \bigoplus_{k=1}^{s-1} R(-sm_0 + kd - \sum r_i d) \right) \\ &= \bigoplus_{k=1}^{s-1} \left( \bigoplus_{1 \leq r_1 < \cdots < r_s \leq n} R(-(sm_0 - kd + \sum r_i d)) \right) \\ &= \bigoplus_{k=1}^{s-1} \left( \bigoplus_{0 \leq r_1 < \cdots < r_s \leq n-1} R(-(sm_0 + (s-k)d + \sum r_i d)) \right) \\ &= \bigoplus_{k=1}^{s-1} \left( \bigoplus_{0 \leq r_1 < \cdots < r_s \leq n-1} R(-(sm_0 + kd + \sum r_i d)) \right) \end{aligned}$$

*Notation 1.3.* Given two integers  $m \geq t \geq 1$ , it will be useful to denote by  $\sigma(m, t)$  the collection (with repetitions) of all possible sums of  $t$  distinct nonnegative integers which are all strictly smaller than  $m$ , i.e.,

$$\sigma(m, t) = \left\{ \sum_{0 \leq r_1 < r_2 < \cdots < r_t \leq m-1} r_i \right\}.$$

For instance,  $\sigma(4, 2) = \{1, 2, 3, 3, 4, 5\}$ . Note that for all  $t$  and  $m$  with  $1 \leq t \leq m$ ,  $\#\sigma(m, t) = \binom{m}{t}$ .

The weighted graded free resolution of  $R/I_2(A)$  given by the Eagon-Northcott complex can now be written as follows:

$$\begin{aligned} 0 \rightarrow \bigoplus_{k=1}^{n-1} R(-(nm_0 + kd + \binom{n}{2}d)) \rightarrow \cdots \rightarrow \bigoplus_{k=1}^{s-1} \left( \bigoplus_{r \in \sigma(n, s)} R(-(sm_0 + kd + rd)) \right) \rightarrow \cdots \\ \cdots \rightarrow \bigoplus_{r \in \sigma(n, 2)} R(-(2m_0 + d + rd)) \rightarrow R \rightarrow R/I_2(A) \rightarrow 0. \end{aligned}$$

**1.3. Mapping cone.** In this section, we will establish our notation for mapping cones, complexes and some facts on mapping cones that we will need.

For any complex  $\mathbf{F} = \oplus F_i$ , denote by  $(\delta_{\mathbf{F}})_i : F_i \rightarrow F_{i-1}$  the boundary maps of  $\mathbf{F}$ , and for any  $t \geq 1$ , let  $\mathbf{F}^t$  be the complex whose  $i$ th term is  $(F^t)_i := F_{i-t}$ . Now if

$\mu : \mathbf{F} \rightarrow \mathbf{G}$  is a map of complexes, the *mapping cone* (or *cone*) over  $\mu$  is the complex  $\mathbf{G} \oplus \mathbf{F}^1$  and it is denoted by  $\text{cone}(\mu)$ . The boundary maps of this complex are

$$\begin{pmatrix} (\delta_{\mathbf{G}})_i & (-1)^i \mu_{i-1} \\ 0 & (\delta_{\mathbf{F}})_{i-1} \end{pmatrix} : G_i \oplus F_{i-1} \rightarrow G_{i-1} \oplus F_{i-2},$$

i.e.,  $(\delta_{\text{cone}(\mu)})_i(g_i, f_{i-1}) = ((\delta_{\mathbf{G}})_i(g_i) + (-1)^i \mu_{i-1}(f_{i-1}), (\delta_{\mathbf{F}})_{i-1}(f_{i-1}))$ .

Let's recall a few well known facts on mapping cones that we will need in the sequel. If  $\mathbf{G}$  is acyclic, i.e.,  $H_i(\mathbf{G}) = 0$  for  $i \geq 1$ , then  $\text{cone}(\mu)$  is exact up to degree 1, i.e.,  $H_i(\text{cone}(\mu)) = 0$  for all  $i \geq 2$ . When  $\mathbf{G}$  is exact and moreover  $\mu_0$  is injective, then  $\text{cone}(\mu)$  is acyclic. A situation of special interest is when  $\mathbf{F}$  is a resolution of  $R/J$  and  $\mathbf{G}$  a resolution of  $R/I$  for two ideals  $I$  and  $J$  in  $R$ . Then, given a map of complexes  $\mu : \mathbf{F} \rightarrow \mathbf{G}$ ,  $\text{cone}(\mu)$  resolves  $R/I + \mu_0(R)$  provided  $\mu_0(J)$  is contained in  $I$ . In particular, consider the following situation: let  $I$  be an ideal in  $R$  and take an element  $z \in R$ . Then,

$$0 \rightarrow R/(I : z) \xrightarrow{\mu} R/I \rightarrow R/I + (z) \rightarrow 0$$

is exact, where  $\mu$  is the map given by multiplication by  $z$ . Now if  $\mathbf{F}$  resolves  $R/(I : z)$ ,  $\mathbf{G}$  resolves  $R/I$ , and  $\mu : \mathbf{F} \rightarrow \mathbf{G}$  is a map of complexes induced by  $\mu$ , then  $\text{cone}(\mu)$  resolves  $R/I + (z)$ .

Let's consider now the graded version of the previous statements. Assume that  $I$  and  $J$  are homogeneous ideals in  $R$ , and consider  $\mathbf{F}$  a graded resolution of  $R/J$ ,  $\mathbf{G}$  a graded resolution of  $R/I$ , and  $\mu : \mathbf{F} \rightarrow \mathbf{G}$  a graded map of complexes with  $\mu(J) \subset I$ . Then, the exact complex  $\text{cone}(\mu)$  is the graded resolution of  $R/I + \mu_0(R)$ . In the particular case where  $J = (I : z)$  for some homogeneous element  $z \in R$  of degree  $\delta$ , the degree zero map  $\mu : R(-\delta) \rightarrow R$  given by multiplication by  $z$  induces a graded map of complexes  $\mu : \mathbf{F}(-\delta) \rightarrow \mathbf{G}$  and  $\text{cone}(\mu)$  is a graded free resolution of  $R/I + (z)$ . Recall that a resolution  $\mathbf{F}$  of  $R/I$  is *minimal* if the ranks of the  $F_i$ 's are minimal or, equivalently, if  $(\delta_{\mathbf{F}}(\mathbf{F})) \subset \mathbf{m}\mathbf{F}$  where  $\mathbf{m} = (x_0, \dots, x_n)$  is the irrelevant maximal ideal. Thus, in a minimal graded resolution, there are no degree zero components in the resolution unless they are identically zero. If  $\mathbf{F}$  and  $\mathbf{G}$  are minimal graded free resolutions of  $R/I : z$  and  $R/I$  respectively, then the only possible obstructions for  $\text{cone}(\mu)$  to be minimal are the degree zero components in  $\mu$ . In other words, if  $\mathbf{F} = \bigoplus \mathbf{F}_i$ ,  $\mathbf{G} = \bigoplus \mathbf{G}_i$  with  $F_i = \bigoplus_j R(-d_{ij})$  and  $G_i = \bigoplus_j R(-c_{ij})$ , and  $\mu : \mathbf{F} \rightarrow \mathbf{G}$  is the graded map of complexes induced by multiplication by  $z$ ,  $R(-\delta) \rightarrow R$ , then  $\text{cone}(\mu)$  is a minimal graded free resolution of  $R/I + (z)$  provided whenever  $d_{ij} = c_{ij}$  for some  $i$  and  $j$ , the projection of the restriction map  $\mu_i|_{R(-d_{ij})}$  onto  $R(-c_{ij})$  is identically zero. If it is not zero, we can split off or cancel the same two summands  $R(-d_{ij})$  from both  $F_i$  and  $G_i$  in the mapping cone construction to achieve minimality.

**Definition 1.4.** The *minimized mapping cone* of the map of complexes  $\mu : \mathbf{F} \rightarrow \mathbf{G}$ , denoted by  $\text{Mincone}(\mu)$ , is the complex obtained from  $\text{cone}(\mu)$  after splitting off all possible summands.

If  $z \in R$  is homogeneous element of degree  $\delta$ ,  $\mathbf{F}$  and  $\mathbf{G}$  are minimal graded free resolutions of  $R/I : z$  and  $R/I$  respectively, and  $\mu : \mathbf{F}(-\delta) \rightarrow \mathbf{G}$  is the graded map of complexes induced by multiplication by  $z$ , then  $\text{Mincone}(\mu)$  is a minimal graded free resolution of  $R/I + (z)$ .

We finally mention the following result that we will need later on.

**Proposition 1.5.** *Let*

$$\mathbf{F} : 0 \rightarrow F_s \xrightarrow{f_s} F_{s-1} \xrightarrow{f_{s-1}} \cdots \rightarrow F_t \xrightarrow{f_t} \cdots \rightarrow F_1 \xrightarrow{f_1} F_0$$

and

$$\mathbf{G} : 0 \rightarrow G_s \xrightarrow{g_s} G_{s-1} \xrightarrow{g_{s-1}} \cdots \rightarrow G_t \xrightarrow{g_t} \cdots \rightarrow G_1 \xrightarrow{g_1} G_0$$

be two exact complexes of free modules and  $\varphi = \oplus \varphi_t : \mathbf{F} \rightarrow \mathbf{G}$  be a map of complexes. Suppose that the dual  $\mathbf{F}^*$  is exact. If  $\varphi_s = 0$  then we have a homotopy to  $\varphi$  given by  $\psi_i : F_{i-1} \rightarrow G_i$  for  $1 \leq i \leq s$  with  $\psi_s = 0$  and  $\varphi_i = g_{i+1} \circ \psi_{i+1} + (-1)^{s-i} \psi_i \circ f_i$ . In particular,  $\varphi_0(F_0) \subset g_1(G_1)$ .

*Proof.* Let  $\psi_s : F_{s-1} \rightarrow G_s$  be the zero map. Since,  $\varphi_s = 0$ , we get,  $\varphi_{s-1} \circ f_s = 0$  and hence  $f_s^*(\varphi_{s-1}^*) = 0$ . By the exactness of the dual of  $\mathbf{F}$ , there exist a map  $\psi_{s-1}^* : G_{s-1}^* \rightarrow F_{s-2}^*$  such that  $f_{s-1}^* \circ \psi_{s-1}^* = \varphi_{s-1}^*$ . So, we get,  $\varphi_{s-1} = \psi_{s-1} \circ f_{s-1} - g_s \circ \psi_s$ . Suppose that we have constructed  $\psi_t$  such that,  $\varphi_i = g_{i+1} \circ \psi_{i+1} + (-1)^{s-i} \psi_i \circ f_i$  for all  $i \geq t$ . Now, since  $\varphi$  is a map of complexes, we have  $\varphi_{t-1} \circ f_t = g_t \circ \varphi_t$ . Substituting, we get  $\varphi_{t-1} \circ f_t = g_t \circ (g_{t+1} \circ \psi_{t+1} + (-1)^{s-t} \psi_t \circ f_t) = (-1)^{s-t} g_t \circ \psi_t \circ f_t$ . Thus  $(\varphi_{t-1} + (-1)^{s-t+1} g_t \circ \psi_t) \circ f_t = 0$ . That is,  $f_t^*((\varphi_{t-1}^* + (-1)^{s-t+1} \psi_t^* \circ g_t^*)(G_{t-1}^*)) = 0$ . By the exactness of the dual  $\mathbf{F}^*$ , we get a map  $\psi_{t-1}^* : G_{t-1}^* \rightarrow F_{t-2}^*$  such that  $\varphi_{t-1}^* + (-1)^{s-t+1} \psi_t^* \circ g_t^* = f_{t-1}^* \circ \psi_{t-1}^*$  and hence  $\varphi_{t-1} = g_t \circ \psi_t + (-1)^{s-t+1} \psi_{t-1} \circ f_{t-1}$ . Thus we prove the existence of  $\psi_i$  for all  $i$ . Looking at  $\psi_1 : F_0 \rightarrow G_1$ , we get,  $\varphi_0(F_0) = g_1 \circ \psi_1(F_0) \subset g_1(G_1)$ .  $\square$

## 2. GORENSTEIN IDEALS

In this section we deal with the case when  $k[\Gamma(\mathbf{m})]$  is Gorenstein separately and see that an explicit construction of the minimal free resolution is obtained from one single mapping cone construction using the fact observed in Section 1.1 that  $\mathcal{P} = I_2(A) + I_2(B)$ . Note that the Gorenstein case will also fit into the general construction of iterated mapping cone given in Section 3 as we shall see in Remark 4.5. The proof we provide in this section completes the argument presented in [3, Theorem 2.6] for the case  $n = 4$ .

The explicit computation of the Cohen-Macaulay type of  $k[\Gamma(\mathbf{m})]$  in [9, Corollary 6.2] under the more general assumption of almost arithmetic sequences implies that if  $\mathbf{m}$  is an arithmetic sequence then  $k[\Gamma(\mathbf{m})]$  is Gorenstein if and only if  $b = 2$ .

So let's assume that  $m_0 \equiv 2 \pmod n$  and write  $m_0 = an + 2$ . Then,

$$B = \begin{pmatrix} X_n^a & X_0 & \cdots & X_{n-2} \\ X_0^{a+d} & X_2 & \cdots & X_n \end{pmatrix}.$$

The ideals  $I_2(A)$  and  $I_2(B)$  are both of height  $n - 1$  and the interesting fact is that the ideal  $\mathcal{P} = I_2(A) + I_2(B)$  is Gorenstein of height  $n$ .

We will construct a resolution for  $R/\mathcal{P}$  as follows. We start with the following preliminary lemma. Consider the map

$$(2.1) \quad \alpha : D_{n-2}G^* \cong R^{n-1} \longrightarrow R, \quad g_1^{(i)} g_2^{(n-2-i)} \mapsto (-1)^i [X_n^a X_{i+2} - X_0^{a+d} X_i].$$

**Lemma 2.1.** *Let  $\mathbf{E}$  be the Eagon-Northcott complex which is a minimal resolution of  $R/I_2(A)$  and  $\mathbf{E}^*$  be as in Section 1.2. Then the map  $\alpha : E_{n-1}^* \rightarrow R$  defined in (2.1) induces a map of complexes  $\alpha : \mathbf{E}^* \rightarrow \mathbf{E}$ .*

*Proof.* Consider the basis element  $e_t \otimes g_1^{(i)} g_2^{(n-3-i)}$  in  $\wedge^{n-1} F^* \otimes D_{n-3} G^*$  where  $e_t$  denotes  $((-1)^{n-i} e_1^* \wedge \cdots \wedge e_{t-1}^* \wedge e_{t+1}^* \wedge \cdots \wedge e_n^*)$ . Then

$$\begin{aligned} \alpha(d_{n-1}^*(e_i \otimes g_1^{(r)} g_2^{(s)})) &= (-1)^{i+1} X_{t-1} [X_n^a X_{i+3} - X_0^{a+d} X_{i+1}] + (-1)^i X_t [X_n^a X_{i+2} - X_0^{a+d} X_i] \\ &= (-1)^{i+1} (X_n^a [X_{t-1} X_{i+3} - X_t X_{i+2}] + X_0^{a+d} [X_{t-1} X_{i+1} - X_t X_i]) \\ &\in I_2(A). \end{aligned}$$

So, the composition  $\wedge^{n-1} F^* \otimes D_{n-3} G^* \rightarrow D_{n-2} G^* \xrightarrow{\alpha} R \rightarrow R/I_2(A)$  is zero. Since  $E$  is exact, we can lift the map  $\alpha$  to  $\alpha : E^* \rightarrow E$  as a map of complexes.  $\square$

**Theorem 2.2.** *Let  $\mathbf{m} = (m_0, \dots, m_n)$  be an arithmetic sequence with  $m_0 \equiv 2 \pmod n$ . If  $\mathcal{P} \subset R$  is the defining ideal of the monomial curve  $C_{\mathbf{m}} \subset \mathbb{A}_k^{n+1}$  associated to the sequence  $\mathbf{m}$ , then  $R/\mathcal{P}$  is Gorenstein of codimension  $n$  with minimal resolution given by  $\mathbf{E} \oplus (\mathbf{E}^*)^1$  where  $E$  is the Eagon-Northcott resolution of  $R/I_2(A)$ .*

*Proof.* As we already recalled in Section 1,  $\mathcal{P} = I_2(A) + I_2(B)$ , and  $\mathbf{E} \rightarrow R/I_2(A) \rightarrow 0$  is a minimal resolution of  $R/I_2(A)$ . Let  $\mathbf{E} : 0 \rightarrow R \rightarrow E_1^* \rightarrow \cdots \rightarrow E_{n-2}^* \rightarrow E_{n-1}^*$  be its dual. If  $\alpha : R^{n-1} = E_{n-1}^* \rightarrow R$  is the map defined in (2.1), one has by Lemma 2.1 that  $\alpha$  induces a map of complexes  $\alpha : \mathbf{E}^* \rightarrow \mathbf{E}$ . Hence the mapping cone  $\mathbf{E} \oplus (\mathbf{E}^*)^1$  is exact. Note that the image of  $\alpha$  is the ideal generated by the  $n-1$  principal  $2 \times 2$  minors of  $B$  and  $H_0(\mathbf{E} \oplus \mathbf{E}^*) = \text{Im}(d_1) + \text{Im}(\alpha) = I_2(A) + I_2(B)$  because the rest of the minors of  $B$  are already in the ideal  $I_2(A)$ . So,  $\mathbf{E} \oplus (\mathbf{E}^*)^1$  resolves  $R/\mathcal{P}$  and it is minimal because  $\alpha$  has positive degree.  $\square$

Since for  $i \in [1, n-1]$ , the rank of the free module  $E_i$  is  $i \binom{n}{i+1}$ , we immediately get the Betti numbers of  $R/\mathcal{P}$ :

**Corollary 2.3.** *Let  $\mathbf{m} = (m_0, \dots, m_n)$  be an arithmetic sequence with  $m_0 \equiv 2 \pmod n$ . Then the Betti numbers of  $R/\mathcal{P}$ , where  $\mathcal{P} \subset k[X_0, \dots, X_n]$  is the defining ideal of the monomial curve  $C_{\mathbf{m}}$  associated to the sequence  $\mathbf{m}$ , are  $\beta_0 = 1$ ,  $\beta_i = i \binom{n}{i+1} + (n-i) \binom{n}{i-1}$  for  $i \in [1, n-1]$ , and  $\beta_n = 1$ .*

### 3. EXPLICIT CONSTRUCTION OF A MINIMAL GRADED RESOLUTION

Let's go back to the general situation:  $m_0 = an + b$  with  $a, b$  positive integers and  $b \in [1, n]$ . In this section, we will not really use that  $\mathcal{P} = I_2(A) + I_2(B)$  as in the Gorenstein case but essentially that  $\mathcal{P}$  is minimally generated by the 2 by 2 minors of  $A$  and the principal minors of  $B$ , i.e.,  $\Delta_i := \Delta_{1, i-b+2}(B) = X_n^a X_i - X_0^{a+d} X_{i-b}$  for  $i \in [b, n]$ . In other words, if one sets

$$\forall i \in [b, n], I_i := I_2(A) + (\Delta_b, \dots, \Delta_i),$$

then  $\mathcal{P} = I_n$ . For all  $i \in [b, n]$ , a graded free resolution of  $R/I_i$  will be obtained by a series of iterated mapping cones. Set

$$\delta_i := \deg \Delta_i = m_0(a + d + 1) + (i - b)d, \forall i \in [b, n]$$

(of course the degrees are with respect to the weighted grading). The following lemma is key to the construction of the minimal resolutions.

**Lemma 3.1.** (1)  $I_2(A) : \Delta_b = I_2(A)$ .  
(2)  $\forall i \in [b, n-1], (X_0, X_1, \dots, X_{n-1}) \subseteq I_i : \Delta_{i+1}$ .



*Proof.* (1) holds because  $I_2(A)$  is prime and  $\Delta_b \notin I_2(A)$ . Moreover, for any  $i \in [b, n-1]$  and  $j \in [0, n-1]$ , one has that

$$\begin{aligned} X_j \Delta_{i+1} &\equiv X_n^a X_i X_{j+1} - X_0^{a+d} X_{i-b+1} X_j \pmod{(X_j X_{i+1} - X_{j+1} X_i)} \\ &\equiv X_0^{a+d} X_{i-b} X_{j+1} - X_0^{a+d} X_{i-b+1} X_j \pmod{\Delta_i} \\ &= X_0^{a+d} (X_{i-b} X_{j+1} - X_{i-b+1} X_j) \end{aligned}$$

which implies that  $X_j \Delta_{i+1} \in I_2(A) + (\Delta_i)$  because  $X_k X_{l+1} - X_{k+1} X_l \in I_2(A)$  for all  $k, l \in [0, n-1]$ , and we are done.  $\square$

*Remark 3.2.* Observe that Lemma 3.1 (2) implies that, for all  $i \in [b, n-1]$ , either  $I_i : \Delta_{i+1} = (X_0, X_1, \dots, X_{n-1})$  or  $I_i : \Delta_{i+1} = (X_0, X_1, \dots, X_{n-1}, X_n^\ell)$  for some  $\ell \geq 1$ . Indeed, we will see in (2) of Inductive Step 3.5 that the latter never occurs.

We are now ready for our iterated mapping cone construction. Recall from Section 1.2 that the minimal graded free resolution  $\mathbf{E} = \bigoplus_{i=0}^{n-1} E_i$  of  $R/I_2(A)$  given by the Eagon-Northcott complex of the matrix  $A$  is

$$0 \rightarrow E_{n-1} \rightarrow \dots \rightarrow E_1 \rightarrow E_0 \rightarrow R/I_2(A) \rightarrow 0$$

$$\text{with } E_0 = R \text{ and } E_{s-1} = \bigoplus_{k=1}^{s-1} \left( \bigoplus_{r \in \sigma(n,s)} R(-(sm_0 + kd + rd)) \right) \text{ for all } s \in [2, n].$$

Let  $\mathbf{C}_b = \mathbf{E} \oplus \mathbf{E}^1(-\delta_b)$  denote the mapping cone of the map  $\Delta_b : \mathbf{E} \rightarrow \mathbf{E}$  which is induced by multiplication by  $\Delta_b$  and  $\delta_b = \deg(\Delta_b)$ . By Lemma 3.1 (1) together with the fact that all the individual maps in  $\Delta_b$  are multiplication by  $\Delta_b$  and hence not zero (in fact injective), we get that  $\mathbf{C}_b$  is the minimal resolution of  $R/I_b$ .

*Notation 3.3.* Set  $L(s, k) := \bigoplus_{r \in \sigma(n,s)} R(-[m_0(a + d + s + 1) + kd + rd])$  for all  $s \in [1, n]$  and  $k \geq 1$ . Then, for all  $s \in [2, n]$ ,  $(\mathbf{C}_b)_s = E_s \oplus \left( \bigoplus_{k=1}^{s-1} L(s, k) \right)$ , and the free modules in  $\mathbf{C}_b$  are

$$\begin{aligned} (\mathbf{C}_b)_0 &= E_0 = R, \\ (\mathbf{C}_b)_1 &= E_1 \oplus E_0(-\delta_b) = E_1 \oplus R(-(m_0(a + d + 1))), \\ (\mathbf{C}_b)_s &= E_s \oplus E_{s-1}(-\delta_b) \\ (3.1) \quad &= E_s \oplus \left( \bigoplus_{k=1}^{s-1} L(s, k) \right), \quad \forall s \in [2, n-1] \\ (\mathbf{C}_b)_n &= \bigoplus_{k=1}^{n-1} R(-[m_0(a + d + n + 1) + kd + \binom{n}{2}d]). \end{aligned}$$

*Remark 3.4.* If  $b = n$ ,  $\mathbf{C}_b$  is the minimal resolution of  $R/\mathcal{P}$ . This is indeed the resolution obtained in [3, Theorem 3.4 and Corollary 3.5] in the case  $b = n$ .

Let  $\mathbf{K} = \bigoplus_{i=0}^n K_i$  be the Koszul resolution of  $R/(X_0, \dots, X_{n-1})$ , i. e.,

$$0 \rightarrow K_n \rightarrow \dots \rightarrow K_1 \rightarrow K_0 \rightarrow R/(X_0, \dots, X_{n-1}) \rightarrow 0$$

with  $K_0 = R$ ,  $K_1 = \bigoplus_{k=0}^{n-1} R(-(m_0 + kd))$ , and  $K_s = \bigoplus_{r \in \sigma(n,s)} R(-(sm_0 + rd))$  for all  $s \in [2, n]$ . Note that for all  $s \in [2, n]$  and  $i \in [b, n]$ ,

$$K_s(-\delta_i) = K_s(-(m_0(a + d + 1) + (i - b)d)) = L(s, i - b).$$

For  $i \in [b, n]$ , we construct inductively two sequences of complexes  $\mathbf{C}_i$  and  $\mathbf{M}_i$  both resolving  $R/I_i$  with  $\mathbf{M}_i$  being a minimal resolution. For  $i = b$ ,  $\mathbf{C}_b = \mathbf{M}_b =$

$\mathbf{E} \oplus \mathbf{E}^1(-\delta_b)$  is given in (3.1). We will now prove the following sequence of steps that forms the  $i$ -th step of our construction.

- Inductive Step 3.5.** (1) The minimal resolution of  $R/I_{i-1}$  has length  $n$ .  
 (2)  $I_{i-1} : \Delta_i = (X_0, X_1, \dots, X_{n-1})$ .  
 (3) Multiplication by  $\Delta_i$  on  $R$  induces a map of complexes  $\Delta_i : \mathbf{K}(-\delta_i) \rightarrow \mathbf{M}_{i-1}$ .  
 (4)  $\mathbf{C}_i = \text{cone}(\Delta_i)$  resolves  $R/I_i$ .  
 (5)  $\mathbf{M}_i$  is the minimized mapping cone of  $\Delta_i$  and is given by
- $(\mathbf{M}_i)_0 = R$ ,
  - $(\mathbf{M}_i)_1 = E_1 \oplus \bigoplus_{k=0}^{i-b} R(-[m_0(a+d+1) + kd])$ ,
  - $(\mathbf{M}_i)_s = \begin{cases} E_s \oplus \left( \bigoplus_{k=s-1}^{i-b} L(s-1, k) \right) & \text{for } s \in [2, i-b+1], \\ E_s \oplus \left( \bigoplus_{k=i-b+1}^{s-1} L(s, k) \right) & \text{for } s \in [i-b+2, n]. \end{cases}$
- (6) The minimal resolution of  $R/I_i$  has length  $n$ .

(1)  $\Rightarrow$  (2). As observed in Remark 3.2, by Lemma 3.1 (2) one has that  $I_{i-1} : \Delta_i$  is either  $(X_0, X_1, \dots, X_{n-1})$  or  $(X_0, X_1, \dots, X_{n-1}, X_n^\ell)$  for some  $\ell \geq 1$ , and it is well-known that the Koszul complex provides minimal graded free resolutions for  $R/(X_0, \dots, X_{n-1})$  and  $R/(X_0, \dots, X_{n-1}, X_n^\ell)$ . Consider the Koszul resolution  $\mathbf{K}' = \bigoplus_{i=0}^{n+1} K'_i$  of  $R/(X_0, \dots, X_{n-1}, X_n^\ell)$ :

$$0 \rightarrow K'_{n+1} \rightarrow \dots \rightarrow K'_1 \rightarrow K'_0 \rightarrow R/(X_0, \dots, X_{n-1}, X_n^\ell) \rightarrow 0.$$

Assume that  $I_{i-1} : \Delta_i = (X_0, X_1, \dots, X_{n-1}, X_n^\ell)$  for some  $\ell \geq 1$  and consider the complex map  $\Delta'_i : \mathbf{K}'(-\delta_i) \rightarrow \mathbf{M}_{i-1}$  induced by multiplication by  $\Delta_i$ . Then the mapping cone of  $\Delta'_i$  provides a free resolution of  $R/I_i$  and since  $\mathbf{K}'(-\delta_i)$  and  $\mathbf{M}_{i-1}$  have length  $n+1$  and  $n$  respectively,  $\text{cone}(\Delta'_i)$  has length  $n+2$ . It may not be minimal nevertheless, since  $(\Delta'_i)_{n+1} : K'_{n+1}(-\delta_i) \rightarrow 0$  is the zero map, no cancelation can occur at the last step of  $\text{cone}(\Delta'_i)$  and  $R/I_i$  would have a minimal resolution of length  $n+2$  which is impossible since  $R$  is a regular ring of length  $n+1$ . Thus,  $I_{i-1} : \Delta_i = (X_0, X_1, \dots, X_{n-1})$ .

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) and (5)  $\Rightarrow$  (6) are straightforward. It remains to show that (4)  $\Rightarrow$  (5).

Set  $i = b + t$ . The complex map  $\Delta_{b+t} : \mathbf{K}(-\delta_{b+t}) \rightarrow \mathbf{M}_{b+t-1}$  induced by multiplication by  $\Delta_{b+t}$  (which is of degree  $\delta_{b+t} = m_0(a+d+1) + td$ ) is given by the following diagram (the left column is the shifted Koszul complex  $\mathbf{K}(-\delta_{b+t})$  that resolves  $R(-[m_0(a+d+1) + td])/(X_0, \dots, X_{n-1})$  minimally, and the column on the right hand side is  $\mathbf{M}_{b+t-1}$  that resolves  $R/I_{b+t-1}$  minimally):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
L(n, t) & \xrightarrow{(\Delta_{b+t})_n} & \bigoplus_{k=t}^{n-1} L(n, k) \\
\downarrow & & \downarrow \\
L(n-1, t) & \xrightarrow{(\Delta_{b+t})_{n-1}} & E_{n-1} \oplus \left( \bigoplus_{k=t}^{n-2} L(n-1, k) \right) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
L(t+2, t) & \xrightarrow{(\Delta_{b+t})_{t+2}} & E_{t+2} \oplus \left( \bigoplus_{k=t}^{t+1} L(t+2, k) \right) \\
\downarrow & & \downarrow \\
L(t+1, t) & \xrightarrow{(\Delta_{b+t})_{t+1}} & E_{t+1} \oplus L(t+1, t) \\
\downarrow & & \downarrow \\
L(t, t) & \xrightarrow{(\Delta_{b+t})_t} & E_t \oplus L(t-1, t-1) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
L(3, t) & \xrightarrow{(\Delta_{b+t})_3} & E_3 \oplus \left( \bigoplus_{k=2}^{t-1} L(2, k) \right) \\
\downarrow & & \downarrow \\
L(2, t) & \xrightarrow{(\Delta_{b+t})_2} & E_2 \oplus \left( \bigoplus_{k=1}^{t-1} L(1, k) \right) \\
\downarrow & & \downarrow \\
L(1, t) & \xrightarrow{(\Delta_{b+t})_1} & E_1 \oplus \bigoplus_{k=0}^{t-1} R(-[m_0(a+d+1) + kd]) \\
\downarrow & & \downarrow \\
R(-[m_0(a+d+1) + td]) & \xrightarrow{(\Delta_{b+t})_0} & R
\end{array}$$

Now observe in the previous diagram that, for  $s \in [t+1, n]$ , the left side  $(\mathbf{K}(-\delta_{b+t}))_s$  corresponds to the summand  $k = t$  on the right hand side and we will show that it splits off entirely.

The following lemma shows that none of these maps are identically zero.

**Lemma 3.6.** *Let  $\mathbf{K}$ ,  $\mathbf{M}$  and  $\Delta_{b+t} : \mathbf{K}(-\delta_{b+t}) \rightarrow \mathbf{M}_{b+t-1}$  be as before. Then,  $(\Delta_{b+t})_i \neq 0$  for all  $0 \leq i \leq n$ . In fact, if we choose a basis for the free modules  $K_i$ ,*

then we can pick a map of complexes  $\Delta_{b+t}$  such that  $(\Delta_{b+t})_i$  is not zero on any of the chosen basis elements of  $K_i$ .

*Proof.* Since  $\mathbf{M}$  is minimal of length  $n$  and  $(\Delta_{b+t})_0$  given by multiplication by  $\Delta_{b+t}$  is injective (not zero), all the maps  $(\Delta_{b+t})_i, 0 \leq i \leq n-1$  are not zero and can be so chosen to be not zero on any chosen basis elements of  $K_t$ . The only question is for the last map  $(\Delta_{b+t})_n$ . This we will take care by the Proposition 1.5. Since  $\Delta_{b+t}$  is not contained in the ideal  $I_{b+t-1}$ , we get that  $(\Delta_{b+t})_0(R) = \Delta_{b+t}R$  is not contained in the image of  $\mathbf{M}_1$ . Thus by Proposition 1.5, we see that  $(\Delta_{b+t})_n$  is not equal to zero. Since  $K_n = R$ , therefore, we get that  $(\Delta_{b+t})_n$  is not equal to zero on a basis for the free module  $K_n$ .  $\square$

Next we show that the projection of  $(\Delta_{b+t})_s$  onto  $L(s, t)$  is not zero for any  $s \geq t+1$ . By degree consideration, the projection of  $(\Delta_{b+t})_s$  onto  $L(s, k)$  for  $k > t$  is certainly zero. What is left follows from the next lemma.

**Lemma 3.7.** *For all  $1 \leq t \leq n-b$ , the projection of  $(\Delta_{b+t})_k(R(-[m_0(a+d+k+1)+td+rd]))$  onto  $R(-[m_0(a+d+k+1)+td+rd])$  is not zero, and hence is a multiplication by a unit, for every  $k \in [t+1, n]$  and  $r \in \sigma(n, k)$ .*

*Proof.* It suffices to show that

$$(3.2) \quad (\Delta_{b+t})_k(R(-[m_0(a+d+k+1)+td+rd])) \not\subset E_k$$

that is, the projection of  $(\Delta_{b+t})_k(L(k, t))$  onto  $L(k, t)$  is not zero. If this projection is not zero for some  $k$  and  $t$ , then the projection of  $(\Delta_{b+t})_k(R(-[m_0(a+d+k+1)+td+rd]))$  onto  $R(-[m_0(a+d+k+1)+td+rd])$  is not zero for the lowest  $r \in \sigma(n, k)$ . Then we can split it off and go to the next smallest  $r$  and so we get the lemma.

Now the rest of the proof is by descending induction on  $k$ . None of these maps  $\Delta$  are identically zero by Lemma 3.6. If  $k = n$ ,  $E_n = 0$  and (3.2) holds, hence the lemma is true for all  $t$ .

Assume that the lemma holds for all  $k \in [s+1, n]$  and for some  $s$ . Let  $k = s$ . Suppose that there is an  $r = r_1 + \dots + r_s$  such that  $(\Delta_{b+t})_s(R(-[m_0(a+d+s+1)+td+rd]))$  is entirely in  $E_s$ . Pick  $r_0$  to be the smallest non negative integer not in  $\{r_1, \dots, r_s\}$ . Consider the commutative diagram:

$$\begin{array}{ccc} R(-[m_0(a+d+s+2)+td+r+r_0d]) & \longrightarrow & (\mathbf{M}_{b+t-1})_{s+1} \\ \downarrow & & \downarrow \\ L(s, t) & \longrightarrow & (\mathbf{M}_{b+t-1})_s \end{array}$$

Since the lemma is true for  $s+1$ , we can take for every  $r \in \sigma(n, s+1)$ ,

$$(\Delta_{b+t})_{s+1}(R(-[m_0(a+d+s+2)+td+rd+r_0d])) = R(-[m_0(a+d+s+2)+td+rd+r_0d]) + \dots$$

Continuing with the vertical arrow on the right, we see that  $R(-[m_0(a+d+s+2)+td+rd+r_0d])$  maps onto

$$\Delta_{b+t-1}E_s + \sum_{i \geq 0} (\pm) X_{r_i} R(-[m_0(a+d+s+2)+td+(r+r_0-r_i)d]).$$

Thus every one of the  $r_i$  and in particular  $X_{r_0}$  that makes up the sum  $r+r_0$  will appear as part of the image in  $(M_{b+t-1})_s/E_s$ . Thus this image is not contained in  $E_s \oplus (X_{r_1}, X_{r_2}, \dots, X_{r_s}) \oplus R(-[m_0(a+d+s+2)+(t-1)d+(r+r_0-r_i)d])$ . If we first come down and then apply  $(\Delta_{b+t-1})_s$ , the image is  $(\Delta_{b+t-1})_s(\sum_{i \geq 0} \pm X_{r_i} 1(-[m_0(a+d+s+2)+td+(r+r_0-r_i)d]))$  which is contained in  $(E_s \oplus (X_{r_1}, X_{r_2}, \dots, X_{r_s}) \oplus R(-[m_0(a+d+s+2)+td+(r+r_0-r_i)d]))$ .

$d + s + 2) + td + (r + r_0 - r_i)d]$  - a contradiction. This completes the induction and the proof.  $\square$

For  $s \in [t + 1, n]$ , the left side  $(\mathbf{K}(-\delta_{b+t}))_s$  splits off entirely with the summand  $k = t$  on the right hand side. On the other hand, for  $s \in [0, t]$ , no cancelation occurs. So the minimal free resolution of  $R/I_i$  is as described in (5) of Inductive Step 3.5. This completes our inductive construction and we have proved the following:

**Theorem 3.8.** *Let  $\mathbf{m} = (m_0, \dots, m_n)$  be an arithmetic sequence with common difference  $d$  and write  $m_0 = an + b$  for  $a, b$  two positive integers with  $b \in [1, n]$ . Consider the polynomial ring  $R = k[X_0, \dots, X_n]$  with  $\deg X_i = m_i$ . Let  $J \subset R$  be the defining ideal of the rational normal curve and  $\mathbf{E}$  be its minimal resolution given by the Eagon-Northcott complex. Set  $\Delta_i := X_n^a X_i - X_0^{a+d} X_{i-b}$  for all  $i \in [b, n]$ ,  $\delta_i := \deg \Delta_i = m_0(a+d+1) + (i-b)d$ , and consider the ideal  $I_i := J + (\Delta_b, \dots, \Delta_i) \subset R$ . Then for all  $i \in [b, n]$ ,  $R/I_i$  is Cohen-Macaulay of codimension  $n$  with minimal graded free resolution  $\mathbf{M}_i$  given by*

$$\begin{aligned} \mathbf{M}_b &= \text{Mincone}(\Delta_b : \mathbf{E}(-\delta_b) \rightarrow \mathbf{E}) = \text{cone}(\Delta_b : \mathbf{E}(-\delta_b) \rightarrow \mathbf{E}) \\ \mathbf{M}_i &= \text{Mincone}(\Delta_i : \mathbf{K}(-\delta_i) \rightarrow \mathbf{M}_{i-1}), \quad \forall i \in [b+1, n] \end{aligned}$$

where  $\mathbf{K}$  is the Koszul resolution of  $(X_0, \dots, X_{n-1})$ . The free modules in  $\mathbf{M}_i$  are explicitly given by

$$0 \rightarrow (\mathbf{M}_i)_n \rightarrow (\mathbf{M}_i)_{n-1} \rightarrow \dots \rightarrow (\mathbf{M}_i)_1 \rightarrow R \rightarrow R/I_i \rightarrow 0$$

where

$$\begin{aligned} \bullet (\mathbf{M}_i)_0 &= R, \\ \bullet (\mathbf{M}_i)_1 &= E_1 \oplus \bigoplus_{k=0}^{i-b} R(-[m_0(a+d+1) + kd]), \\ \bullet (\mathbf{M}_i)_s &= \begin{cases} E_s \oplus \left( \bigoplus_{k=s-1}^{i-b} L(s-1, k) \right) & \text{for } s \in [2, i-b+1], \\ E_s \oplus \left( \bigoplus_{k=i-b+1}^{s-1} L(s, k) \right) & \text{for } s \in [i-b+2, n]. \end{cases} \end{aligned}$$

$$\text{In particular, } (\mathbf{M}_i)_n = \begin{cases} L(n-1, n-1) & \text{if } i = n \text{ and } b = 1, \\ \left( \bigoplus_{k=i-b+1}^{n-1} L(n, k) \right) & \text{otherwise.} \end{cases}$$

**Corollary 3.9.** *Using notations in Theorem 3.8, the Cohen-Macaulay of type of  $R/I_i$  is*

$$\begin{cases} n & \text{if } i = n \text{ and } b = 1, \\ n-1+b-i & \text{otherwise.} \end{cases}$$

In particular, since  $I_n = \mathcal{P}$  is the defining ideal of the monomial curve  $C_{\mathbf{m}}$ , we get the following theorem.

**Theorem 3.10.** *Let  $\mathbf{m} = (m_0, \dots, m_n)$  be an arithmetic sequence and write  $m_0 = an + b$  for  $a, b$  two positive integers with  $b \in [1, n]$ . If  $\mathcal{P} \subset R := k[X_0, \dots, X_n]$  is the defining ideal of the monomial curve  $C_{\mathbf{m}} \subset \mathbb{A}_k^{n+1}$  associated to  $\mathbf{m}$  then  $R/\mathcal{P}$  is Cohen-Macaulay of codimension  $n$  and its minimal graded free resolution  $\mathbf{M}_n$  is*

$$0 \rightarrow F_n \rightarrow E_{n-1} \oplus F_{n-1} \rightarrow \dots \rightarrow E_1 \oplus F_1 \rightarrow R \rightarrow R/\mathcal{P} \rightarrow 0$$

where, for all  $s \in [2, n]$ ,  $E_{s-1} = \bigoplus_{k=1}^{s-1} \left( \bigoplus_{r \in \sigma(n,s)} R(-(sm_0 + kd + rd)) \right)$ , and

$$\begin{aligned} F_1 &= \left( \bigoplus_{k=0}^{n-b} R(-[m_0(a+d+1) + kd]) \right) \\ F_2 &= \left( \bigoplus_{k=1}^{n-b} \left( \bigoplus_{r=0}^{n-1} R(-[(m_0(a+d+2) + kd + rd)]) \right) \right) \\ F_s &= \begin{cases} \left( \bigoplus_{k=0}^{n-b+1-s} \left( \bigoplus_{r \in \sigma(n,s-1)} R(-[m_0(a+d+s) + kd + rd]) \right) \right) & \text{if } s \in [3, n-b+1], \\ \left( \bigoplus_{k=1}^{s-n+b-1} \left( \bigoplus_{r \in \sigma(n,s)} R(-[m_0(a+d+s+1) + kd + rd]) \right) \right) & \text{if } s \in [n-b+2, n]. \end{cases} \end{aligned}$$

#### 4. CONSEQUENCES

The first direct consequence of Theorem 3.10 shows that [3, Conjecture 2.2], which was first stated in [12], holds:

**Theorem 4.1.** *With notations as in Theorem 3.10, the Betti numbers of  $R/\mathcal{P}$  only depend on  $n$  and on the value of  $m_0$  modulo  $n$ .*

*Proof.* If one reads the Betti numbers  $\{\beta_j, j \in [0, n]\}$  in the minimal graded free resolution in Theorem 3.10, one gets that  $\beta_0 = 1$  and

$$(4.1) \quad \beta_j = j \binom{n}{j+1} + \begin{cases} (n-b+2-j) \binom{n}{j-1} & \text{if } 1 \leq j \leq n-b+1, \\ (j-n+b-1) \binom{n}{j} & \text{if } n-b+1 < j \leq n, \end{cases}$$

where  $m_0 \equiv b \pmod n$  and  $b \in [1, n]$  and hence the statement holds.  $\square$

**Example 4.2.** For  $n = 4$ , if  $\mathcal{P} \subset R = k[X_0, \dots, X_4]$  is the defining ideal of the monomial curve associated to an arithmetic sequence  $\mathbf{m} = (m_0, \dots, m_4)$ , the 4 different patterns for the global Betti numbers of  $R/\mathcal{P}$  are as follows. They correspond respectively to  $b = 1, 2, 3$  and 4:

$$\begin{array}{cccccccccccccccc} 0 & \rightarrow & R^4 & \rightarrow & R^{15} & \rightarrow & R^{20} & \rightarrow & R^{10} & \rightarrow & R & \rightarrow & R/\mathcal{P} & \rightarrow & 0 \\ 0 & \rightarrow & R & \rightarrow & R^9 & \rightarrow & R^{16} & \rightarrow & R^9 & \rightarrow & R & \rightarrow & R/\mathcal{P} & \rightarrow & 0 \\ 0 & \rightarrow & R^2 & \rightarrow & R^7 & \rightarrow & R^{12} & \rightarrow & R^8 & \rightarrow & R & \rightarrow & R/\mathcal{P} & \rightarrow & 0 \\ 0 & \rightarrow & R^3 & \rightarrow & R^{11} & \rightarrow & R^{14} & \rightarrow & R^7 & \rightarrow & R & \rightarrow & R/\mathcal{P} & \rightarrow & 0 \end{array}$$

For example, the two arithmetic sequences  $\mathbf{m}_1 = (11, 13, 15, 17, 19)$  and  $\mathbf{m}_2 = (7, 12, 17, 22, 27)$  fit into the third pattern because  $11 \equiv 7 \equiv 3 \pmod 4$ . Denoting by  $\mathcal{P}_1$  and  $\mathcal{P}_2$  the defining ideal of  $C_{\mathbf{m}_1}$  and  $C_{\mathbf{m}_2}$  respectively, the minimal graded free resolutions of  $R/\mathcal{P}_1$  and  $R/\mathcal{P}_2$  are given by Theorem 3.10, and the result can

easily be checked using the softwares CoCoA, Macaulay2 or Singular:

$$\begin{aligned}
0 \rightarrow R(-115) \oplus R(-117) &\longrightarrow \begin{array}{c} R(-58) \oplus R(-60) \\ \oplus R(-62) \oplus R(-98) \\ \oplus R(-100) \oplus R(-102) \\ \oplus R(-104) \end{array} \longrightarrow \begin{array}{c} R(-41) \oplus R(-43)^2 \\ \oplus R(-45)^2 \oplus R(-47)^2 \\ \oplus R(-49) \oplus R(-68) \\ \oplus R(-70) \oplus R(-72) \\ \oplus R(-74) \end{array} \\
&\longrightarrow \begin{array}{c} R(-26) \oplus R(-28) \\ \oplus R(-30)^2 \oplus R(-32) \\ \oplus R(-34) \oplus R(-55) \\ \oplus R(-57) \end{array} \longrightarrow R \longrightarrow R/\mathcal{P}_1 \rightarrow 0. \\
0 \rightarrow R(-117) \oplus R(-122) &\longrightarrow \begin{array}{c} R(-63) \oplus R(-68) \\ \oplus R(-73) \oplus R(-95) \\ \oplus R(-100) \oplus R(-105) \\ \oplus R(-110) \end{array} \longrightarrow \begin{array}{c} R(-41) \oplus R(-46)^2 \\ \oplus R(-51)^2 \oplus R(-56)^2 \\ \oplus R(-61)^2 \oplus R(-66) \\ \oplus R(-71) \oplus R(-76) \end{array} \\
&\longrightarrow \begin{array}{c} R(-24) \oplus R(-29) \\ \oplus R(-34)^2 \oplus R(-39) \\ \oplus R(-44) \oplus R(-49) \\ \oplus R(-54) \end{array} \longrightarrow R \longrightarrow R/\mathcal{P}_2 \rightarrow 0.
\end{aligned}$$

*Remark 4.3.* It is important to note that the phenomenon described in Theorem 4.1 is something special about arithmetic sequences only. In general, if  $0 < m_0 < \dots < m_n$  is a sequence of integers then the Betti numbers of the monomial curve defined by  $x_i = t^{m_i}$  do not depend only on  $n$  and the remainder of  $m_0$  upon division by  $n$ . It does not hold even for almost arithmetic sequences, even in dimension 3. In dimension 3, a monomial curve is either a complete intersection with  $\beta_1 = 2, \beta_2 = 1$  or an ideal of  $2 \times 2$  minors of a  $3 \times 2$  matrix with  $\beta_1 = 3$  and  $\beta_2 = 2$ . Now, it is easy to see that for  $\mathbf{m} = (7, 10, 15)$ ,  $C_{\mathbf{m}}$  is a complete intersection with  $\mathcal{P} = (X_1^3 - X_2^2, X_0^5 - X_1^2 X_2)$  where as for  $\mathbf{m} = (13, 16, 21)$ ,  $C_{\mathbf{m}}$  is not a complete intersection. However both 7 and 13 are odd and hence equal 1 modulo 2.

**Corollary 4.4.** *If  $I_i \subset R$  is as in Theorem 3.8,  $R/I_i$  is Gorenstein if and only if*

- $b = 2, i = n$ ,
- $b = 1, i = n - 1$ , or
- $n = 1$ .

*In particular, we recover the result of [9, Corollary 6.2] that  $R/\mathcal{P}$  is Gorenstein if and only if  $b = 2$ . Moreover,  $R/I_i$  are never level unless they are Gorenstein.*

*Proof.* If  $n = 1$ ,  $I_1 = \mathcal{P}$  is principal and therefore Gorenstein. By Corollary 3.9, the type of  $R/I_i$  is  $n - 1 + b - i$ . Since  $i \neq n$ ,  $n - 1 + b - i \geq b - 1 > 1$  if  $b > 2$  so  $R/I_i$  is Gorenstein if and only if either  $b = 2, i = n$  or  $b = 1, i = n - 1$ . The non-levelness of  $R/I_i$  when it is not Gorenstein follows directly from the degrees in the resolution.  $\square$

*Remark 4.5.* If  $m_0 \equiv 2 \pmod n$ , i.e., if  $R/\mathcal{P}$  is Gorenstein, then by (4.1) one has that  $\beta_0 = \beta_n = 1$  and  $\beta_j = j \binom{n}{j+1} + (n-j) \binom{n}{j-1}$  for all  $j \in [1, n-1]$  which is Corollary 2.3. Note that this result is obtained here by making a resolution minimal which is obtained through iterated mapping cone construction, while it had been obtained in Section 2 by a direct argument.

*Remark 4.6.* If  $m_0 \equiv 1 \pmod n$ , one gets by (4.1) that for all  $j \in [1, n]$ ,  $\beta_j = j \binom{n}{j+1} + (n+1-j) \binom{n}{j-1}$ . Note that in this case the Betti numbers had already

been obtained in [3, Theorem 3.1] where we show that for all  $j \in [1, n]$ ,  $\beta_j = j \binom{n+1}{j+1}$ . One can easily check that both numbers coincide.

**Theorem 4.7.** *With notations as in Theorem 3.10, the Cohen Macaulay type of  $R/\mathcal{P}$  is the unique integer  $c$  in  $[1, n]$  such that  $c \equiv m_0 - 1 \pmod{n}$ .*

*Proof.* The Cohen-Macaulay type of  $R/\mathcal{P}$ ,  $\beta_n$ , is computed by the first formula in (4.1) when  $b = 1$ , and by the second otherwise. Thus,  $\beta_n = \binom{n}{n-1} = n$  if  $b = 1$ , and  $\beta_n = (b-1)\binom{n}{n} = b-1$  otherwise.  $\square$

Putting together Theorem 4.1 and Theorem 4.7, one gets the following:

**Corollary 4.8.** *With notations as in Theorem 3.10, the Cohen-Macaulay type of  $R/\mathcal{P}$  determines all its Betti numbers.*

Note that in the previous corollary, the same result holds if one substitutes the minimal number of generators of  $\mathcal{P}$  for the Cohen-Macaulay type of  $R/\mathcal{P}$ .

*Remark 4.9.* One can also deduce from the minimal graded free resolution in Theorem 3.10, the value of the (weighted) Castelnuovo-Mumford regularity of  $R/\mathcal{P}$  which is indeed

$$\operatorname{reg}(R/\mathcal{P}) = \begin{cases} \binom{n}{2}d + m_0(a+d) + n(m_0-1) & \text{if } b = 1, \\ \left(\binom{n}{2} + b - 1\right)d + m_0(a+d+1) + n(m_0-1) & \text{if } b \geq 2. \end{cases}$$

On the other hand, the Frobenius number  $g(\mathbf{m})$  of the numerical semigroup  $\Gamma(\mathbf{m})$  can be computed using [10, Theorem 3.2.2] and one gets that  $g(\mathbf{m}) = (a-1)m_0 + d(m_0-1)$  if  $b = 1$ , and  $g(\mathbf{m}) = am_0 + d(m_0-1)$  if  $b \geq 2$ . Thus  $\operatorname{reg}(R/\mathcal{P}) - g(\mathbf{m}) = \left(\binom{n}{2} + b\right)d + m_0 + n(m_0-1)$ . In particular, the regularity is always an upper bound for the conductor  $g(\mathbf{m}) + 1$  of the numerical semigroup  $\Gamma(\mathbf{m})$  and it is, in general, much bigger. The above relation between the regularity of the semigroup ring and the Frobenius number of the semigroup can be nicely expressed as follows:

$$\operatorname{reg}(R/\mathcal{P}) = g(\mathbf{m}) + \sum_{i=0}^n m_i - (n-b)d - n.$$

Finally, we use Theorem 4.1 to prove a conjecture of Herzog and Srinivasan on eventual periodicity of Betti numbers of semigroup rings in our context. Given a sequence of positive integers  $\mathbf{m} = (m_0, \dots, m_n)$  and a positive integer  $j$ , denote by  $\mathbf{m} + (j) = \mathbf{m} + (j, j, \dots, j)$ . Herzog and Srinivasan have conjectured the following:

**Conjecture 4.10** (Herzog and Srinivasan). *Let  $\mathbf{m}$  and  $\mathbf{m} + (j)$  be as above.*

- HS1 *The Betti numbers of the semigroup ring  $k[\Gamma(\mathbf{m} + (j))]$  are eventually periodic in  $j$ .*
- HS2 *The number of minimal generators of the defining ideal of the monomial curve  $C_{\mathbf{m} + (j)}$  is eventually periodic in  $j$  with period  $m_n - m_0$ .*
- HS3 *The number of minimal generators of the defining ideal of the monomial curve  $C_{\mathbf{m} + (j)}$  is bounded for all  $j$ .*

They prove the conjecture for  $n = 2$  and A. Marzullo proves it for some cases when  $n = 4$  in [13]. Our Theorem 4.11 proves this periodicity conjecture in its strong form (HS1) for arithmetic sequences.

**Theorem 4.11.** *If  $\mathbf{m} = (m_0, \dots, m_n)$  is an arithmetic sequence and  $\mathbf{m} + (j) = \mathbf{m} + (j, j, \dots, j)$ , then the Betti numbers of  $C_{\mathbf{m} + (j)}$  are eventually periodic in  $j$  with period  $m_n - m_0$ .*



*Proof.* Let  $\mathbf{m} = (m_0, \dots, m_n)$  be an arithmetic sequence and  $j \in \mathbb{N}$ . Since  $(m_0 + j, \dots, m_n + j)$  is in arithmetic progression,  $\gcd(m_0 + j, \dots, m_n + j) = \gcd(m_0 + j, d)$  where  $d$  is the common difference in  $\mathbf{m}$ . We denote by  $\widetilde{\cdot}$  division by  $\gcd(m_0 + j, d)$ . Then  $\widetilde{m_i + j} = \widetilde{m_0 + j} + i\widetilde{d}$  and  $\widetilde{\mathbf{m} + (j)} = (\widetilde{m_0 + j}, \dots, \widetilde{m_n + j})$  has  $\gcd 1$ . Moreover,  $C_{\mathbf{m}+(j)} = C_{\widetilde{\mathbf{m}+(j)}}$ .

We claim that for  $j \geq nd - m_0$ ,  $\widetilde{\mathbf{m} + (j)}$  is always an arithmetic sequence. If  $\widetilde{m_k + j} = \sum_{i \neq k} r_i \widetilde{m_i + j}$  for  $r_i \in \mathbb{N}$ , then  $k\widetilde{d} + \widetilde{m_0 + j} = \sum_{i \neq k} r_i (\widetilde{m_0 + j} + i\widetilde{d})$  and hence  $\widetilde{d}(k - \sum_{i \neq k} ir_i) = (\widetilde{m_0 + j})(\sum_{i \neq k} r_i - 1) \geq n\widetilde{d}$  and this is not possible since  $k - \sum_{i \neq k} ir_i < n$ . Now,

$$\begin{aligned} \widetilde{m_0 + j + nd} &= \widetilde{m_0 + j} + n\widetilde{d} \\ &\equiv \widetilde{m_0 + j} \pmod{n}. \end{aligned}$$

Note that  $\gcd(m_0 + j, d)$  is periodic with period  $d$ . So the Betti numbers of  $C_{\mathbf{m}+(j)}$  are periodic with period  $nd$  for  $j$  large enough.  $\square$

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